

Geometric Intuition and General Relativity, a Practical Case

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Abstract

Most physics students are easily satisfied with the idea that space-time is curved, and introductory accounts usually appeal to our intuition of two-dimensional curved surfaces imbedded in ordinary space. Four-dimensional geometry and topology can be infinitely more complex than that, and this leads to counter-intuitive results.

To illustrate this idea the geometry of Kramer's exact solution for a rotating fluid is analysed in detail. This solution has a wealth of counter-intuitive and apparently contradictory properties that highlight the complexity of General Relativity.

1. Introduction

In 1984, Kramer [3] discovered a very peculiar exact solution to Einstein's equations of General Relativity. Despite several probably unphysical features, the Kramer solution is interesting for a variety of reasons. For a long time now, it has been impossible to find an exact solution to the Einstein equations which describes a physically realistic star. By "physically realistic" one usually means that

- the "star" is a mass of "perfect fluid",
- the fluid rotates about an axis which is an axis of symmetry of the whole solution, and
- space is asymptotically flat and empty away from the star.

Vacuum solutions with the required symmetry are known (Kerr spacetimes) but a rotationally symmetric region including the Kerr black hole have to be "removed" to leave a so-called **external solution**, which must then be matched smoothly to a fluid solution, called an **internal solution**. Many external and internal solutions are known, but usually they cannot be matched smoothly. It is known that solutions exist, but numerical solutions are the only known so far.

2. The Kramer metric

The metric, as given by Kramer, is

$$2m ds^2 = -b \cos \xi (e^{-\eta/2} d\tau - e^{\eta/2} d\sigma)^2 + \frac{d\eta^2}{\eta - 1} + (\eta - 1) d\tau^2 + \frac{e^\eta d\xi^2}{b \cos \xi}.$$

With the changes of variable $\eta = 1 + r^2$ (suggested by [1]), $\tau = 2\theta$, $\sigma = (2/e)\varphi$ and $b = ev$, one obtains

$$\frac{m}{2} ds^2 = -v \cos \xi (e^{-r^2/2} d\theta - e^{r^2/2} d\varphi)^2 + dr^2 + r^2 d\theta^2 + \frac{e^{r^2} d\xi^2}{4v \cos \xi}.$$

The spatial part of the metric suggests the interpretation of (r, θ, ξ) as analogous to cylindrical coordinates, but the $d\theta$ term in the timelike part of the metric apparently disturbs this interpretation. I say

“apparently” because the change of variable $\varphi = \theta + t$ (suggested by the form of the metric at $r \rightarrow 0$) yields

$$\frac{m}{2} ds^2 = -v \cos \xi (2 \sinh(r^2/2) d\theta + e^{r^2/2} dt)^2 + dr^2 + r^2 d\theta^2 + \frac{e^{r^2} d\xi^2}{4v \cos \xi},$$

which for $t = \text{const.}$ gives a spatial axially symmetric geometry in the vicinity of $r = 0$. The metric can alternatively be expressed in terms of φ and t , in which case it takes the form

$$\frac{m}{2} ds^2 = -v \cos \xi (2 \sinh(r^2/2) d\varphi + e^{-r^2/2} dt)^2 + dr^2 + r^2 (d\varphi - dt)^2 + \frac{e^{r^2} d\xi^2}{4v \cos \xi}$$

This is the form of the metric analysed in [3, 1].

This solution describes a **perfect fluid** (i.e., one whose stress–energy tensor is $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$ [4]) with

$$\rho = \frac{m}{2\kappa_0} (3v \cos \xi e^{-r^2} - 3r^2 - 2) \quad \text{and} \quad p = \frac{m}{2\kappa_0} (2 + r^2 - v \cos \xi e^{-r^2}).$$

The four-velocity of the fluid is such that r, ξ and $\varphi = \theta + t$ are constant along flow lines. Note that the metric does not depend on the values of (t, θ, φ) (technically, $\partial_t, \partial_\theta$ and ∂_φ are **Killing fields**). The fact that the four-velocity of the fluid is proportional to a Killing field is interpreted as **rigid rotation** of the fluid.

Which is the “best” of the preceding four expressions for the metric? The answer is that all have their virtues. In particular, the metric in terms of (t, r, θ, ξ) makes it evident that θ is a regular angular coordinate about the rotation axis $r = 0$. Also, because φ is constant along flow lines, the coordinates (t, r, φ, ξ) are useful “comoving” coordinates. We therefore interpret θ as a “background” angle and φ as a “comoving” angle, and then the metric in terms of $(\varphi, r, \theta, \xi)$ starts seeming the most useful of all, and the “time” coordinate loses importance. After all, we know that in General relativity time cannot be given a universal meaning, while θ and φ can both be given an operational definition.

The range of validity of the coordinates can be determined by the physical requirements that

- 1) the signature of the metric is $(-+++)$. If $v \cos \xi > 0$ the ξ coordinate is spacelike and $e^{-r^2/2} d\theta - e^{-r^2/2} d\varphi$ is timelike. At $\xi = \pm\pi/2$ they exchange roles but the signature stays the same, so by analogy with the Schwarzschild solution, we can expect this to be just a coordinate singularity and not a real one.
- 2) the fluid 4-velocity is timelike, or equivalently,

$$v \cos \xi e^{-r^2} > r^2.$$

This already excludes the possibility that $v \cos \xi < 0$, so $|\xi| < \pi/2$.

- 3) the **dominant energy condition** (i.e. that the energy density is positive as measured by any observer) holds, which is equivalent to $\rho > |p|$, or

$$v \cos \xi e^{-r^2} > 1 + r^2.$$

This clearly implies that the fluid four-velocity is timelike.

- 4) The only way to make physical sense of a finite range of validity which does not involve singularities is to “cut” the fluid solution at the surface of zero pressure and extend it by a vacuum solution “outside” that surface. The condition $p = 0$ is equivalent to

$$v \cos \xi e^{-r^2} = 2 + r^2,$$

so this surface exists only for $v > 2$.

Therefore, the range of validity of the coordinates is restricted to the region

$$v \cos \xi e^{-r^2} \geq 2 + r^2.$$

Note that these conditions can only be satisfied if $v > 2$. This means that the Kramer solution has a minimal rotation rate, so the rotation cannot be switched off. This is because the Kramer solution is a

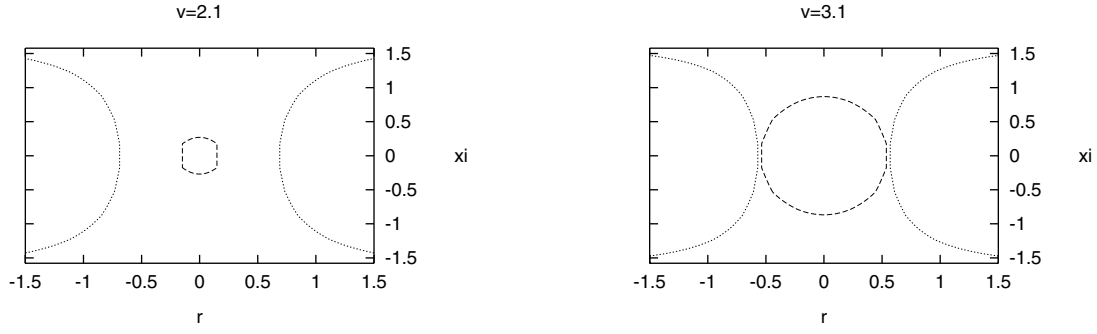
limit, in an ultrarelativistic regime, of a family of solutions which does have a Newtonian limit. This is the first clear indication that the solution might be unphysical, since not only can't the rotation be turned off, but probably it cannot be turned on either! Also, the pressure is negative in this region (i.e., the fluid is under tension), which means that the matter is rather exotic.

In [3] it is also required that in (t, φ) coordinates the ∂_φ killing vector should be spacelike, or equivalently

$$4v \cos \xi \sinh^2(r^2/2) < r^2$$

so that the $t = \text{const}$ sections are spacelike and φ can be interpreted as an angular coordinate throughout its domain of definition. In fact, this is equivalent to requiring that ∂_θ be spacelike in the (t, θ) coordinates. This requirement seems reasonable, but in my opinion it is actually unphysical, because it depends on attaching the role of “absolute time” to the t coordinate.

In [1] it is shown that the range of validity of the coordinates (i.e., the mass of fluid) appears to be convex from the inside, but concave from the outside. More precisely, for large values of the rotation parameter v , the boundary develops a “neck” at the equator, which eventually even “pinches off”; on the other hand, there are always geodesics joining two points of the boundary through the interior of the fluid. This probably prevents the matching of this solution to an asymptotically flat exterior, but this hasn't been proven.



1 The range of validity of the coordinates for two values of the rotation parameter. The oval is the boundary of the fluid, and the outer curve marks the point where the angular coordinates become timelike.

3. Geodesics of the Kramer Metric

Affinely parameterised geodesics are extremal points of the following “action” [2]:

$$\mathcal{S} = \frac{1}{2} \int ds \left\{ -v \cos \xi (e^{-r^2/2} \dot{\theta} - e^{r^2/2} \dot{\varphi})^2 + \dot{r}^2 + r^2 \dot{\theta}^2 + \frac{e^{r^2} \dot{\xi}^2}{4v \cos \xi} \right\}$$

The Euler–Lagrange equations are

$$\begin{cases} v \cos \xi (e^{r^2} \dot{\varphi} - \dot{\theta}) = E & \text{(energy)} \\ E e^{-r^2} + r^2 \dot{\theta} = L & \text{(angular momentum)} \\ \frac{E^2 e^{-r^2} \sin \xi}{v \cos^2 \xi} = \frac{e^{r^2} r \dot{r} \dot{\xi}}{v \cos \xi} + \frac{e^{r^2} \ddot{\xi}}{2v \cos \xi} + \frac{e^{r^2} \sin \xi \dot{\xi}^2}{4v \cos^2 \xi} \\ -2E e^{-r^2} \left(2\dot{\theta} + \frac{E}{v \cos \xi} \right) r + 2r \dot{\theta}^2 + \frac{e^{r^2} r \dot{\xi}^2}{2v \cos \xi} = 2\ddot{r} \end{cases}$$

The third equation can be integrated by multiplying both sides by $e^{r^2} \dot{\xi}$,

$$\begin{cases} v \cos \xi (e^{r^2} \dot{\varphi} - \dot{\theta}) = E \\ E e^{-r^2} + r^2 \dot{\theta} = L \\ \frac{e^{2r^2} \dot{\xi}^2}{4v \cos \xi} - \frac{E^2}{v \cos \xi} = K \\ 2\dot{\theta} (r \dot{\theta} - 2E e^{-r^2} r) + 2K e^{-r^2} r = 2\ddot{r} \end{cases}$$

and the last by multiplying by \dot{r} and using $Ee^{-r^2} + r^2\dot{\theta} = L \Rightarrow 2Ee^{-r^2}r\dot{r} - r\dot{\theta} = r\dot{\theta} + r^2\ddot{\theta}$

$$\begin{cases} v \cos \xi (e^{r^2} \dot{\phi} - \dot{\theta}) = E \\ Ee^{-r^2} + r^2\dot{\theta} = L \\ e^{2r^2} \dot{\xi}^2 = 4(Kv \cos \xi + E^2) \\ \left(\frac{L - Ee^{-r^2}}{r} \right)^2 + Ke^{-r^2} + \dot{r}^2 = S \quad (\text{squared proper length}) \end{cases}$$

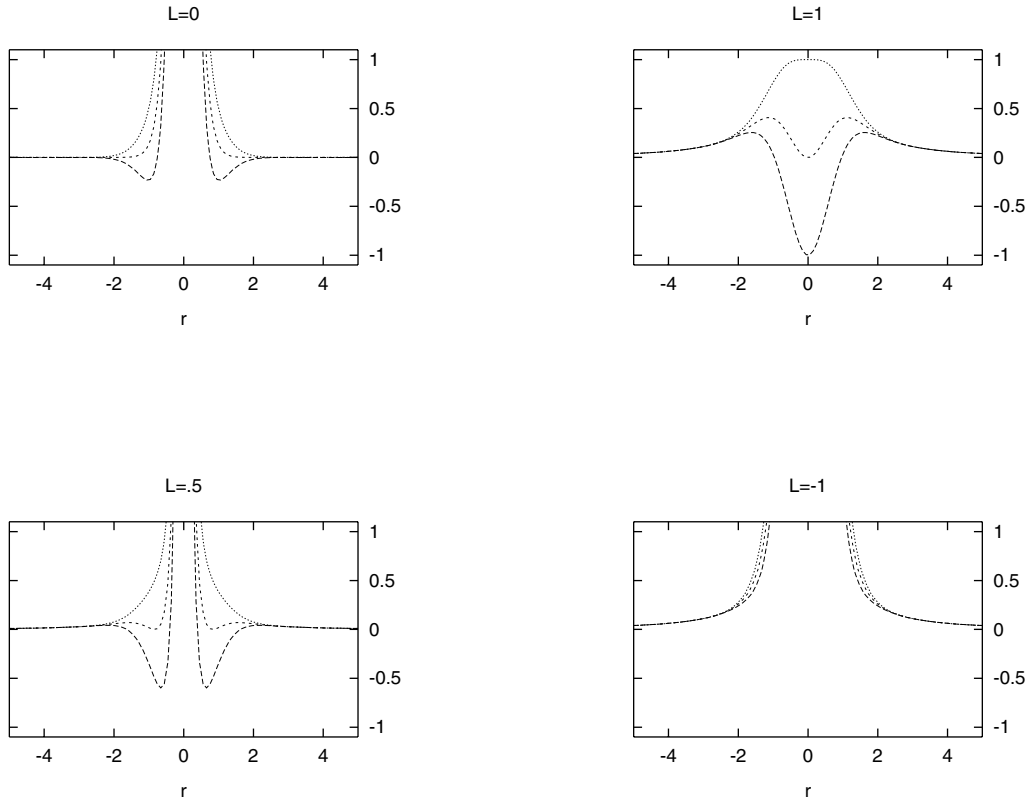
It is rather remarkable that the geodesics of the Kramer metric can be completely reduced to quadratures, and we can use that to our advantage in trying to visualize the geometry.

If $(\phi, r, \theta, \xi)(s)$ is a solution of the affine geodesic equations, the integral S is simply $\frac{1}{2} \int S$. This means that the sign of S determines the character of the geodesic: positive for spacelike geodesics, zero for lightlike and negative for timelike.

The salient feature of these equations is that the variation of r is determined by an effective potential of the form

$$V(r) = \left(\frac{L - Ee^{-r^2}}{r} \right)^2 + Ke^{-r^2}.$$

The “energy” S of this motion is positive for spacelike geodesics, zero for lightlike geodesics and negative for timelike geodesics. This means that, in general, non-spacelike geodesics are bounded in the vicinity of the “axis” $r = 0$. Only in the case $L = 0$ the lightlike geodesics are bounded away from the axis but unbounded in the opposite direction. It is not hard to see that the behaviour of the potential is qualitatively different in the degenerate cases $L = E$, $L = 0 \neq E$, and the generic cases $E/L > 1$ and $E/L < 1$.



2 The effective potential for radial motion, for $E = 1$. The solid line is for $K = 1$, the dotted line for $K = 0$ and the dashed line for $K = -1$.

4. References

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5. Acknowledgements

The author wishes to thank professor Francisco Chinae Trujillo for inspiration, discussion and criticism, and fellow student Francisco Navarro Lérída for discussion. This paper is an outgrowth of a Ph.D. course on “Free-boundary Problems in Gravitation and Self-Gravitating Fluids” taught by professor Chinae in June 2000 at Universidad Complutense de Madrid, Spain.