

# Entropy and Information in Dynamical Systems

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**Abstract:** After briefly recalling the definition and properties of a dynamical system, the concepts of dynamical entropy and information production (positive and negative) are introduced. Dynamical entropy characterizes the average rate of ‘shuffling’ of phase-space by a surjective map  $f: X \rightarrow X$  that describes the dynamics. The examination of dynamics from this point of view reveals a deeper insight into the nature of Chaos.

*“Imagination is more important than knowledge”*

*Albert Einstein: On Science*

## Introduction

Historically, it was always thought that simple systems always exhibit simple behavior, while complex system exhibits (in some way) complex behavior. The celestial mechanics, a playground for classical mechanics, was a very good example of how all our laws of mechanics known till then (XIX.. century) had a practical proof . Henri Poincare’ (1854.-1912.) was the first one to observe that there are systems which do not always behave simply, although there are simple (for a simple system I intend a system which is described by simple equations – planet motion, harmonic oscillator, circular motion ...) Particularly, it was the three-body problem that made the first step into the non-linear physics; it’s obviously a “simple” situation, but it’s ‘solution’ is far from simple. After that, in the beginning of XX.. century there came a fast development in the field of non-linear physics, which took a name Chaos Theory. There is a subtle ambiguity in the concept “non-linear physics” which we have to deal with. All physical phenomena and equations are actually non-linear in their most general form. However, most well-known equations of (for example) classical mechanics, electromagnetism or quantum mechanics are in fact linear (partial) differential equations of some given order. By non-linear phenomena I intend a phenomena which is described by intrinsically non-linear equations, which is not possible to approximate (unless perhaps in some very limited region) by any linear equation. If we define a sequence:  $x_{n+1} = x_n - x_n^2$ , we are obviously dealing with a non-linear phenomena, which has some unexpected aspects, whose appearance as a part of some linear phenomena is not possible. Let’s now reviewsome definitions and properties of dynamical system theory.

## Dynamical Systems and Chaos

When we are speaking about a dynamical system, we are speaking not only about the dynamics as a part of classical mechanics; our definition of dynamics is extended to anything which is evolving with time in some deterministic way - electrical vibrations, chemical reaction, sequence of numbers, even the human population characteristics. Mathematically speaking, dynamical system's "determinism" is the possibility to define a *time evolution operator*  $U_t$  which makes correspondence between states of a given dynamical system. So, if initially we have a state  $x(0) = x_0$ , then at time  $t$  we will have  $x(t) = U_t x_0$ ;  $x(t)$  is completely specified for all  $t$ . Note that in this case for "states" we mean a points in an abstract space, or more precisely, a state in some given suitable "set of states". All becomes clear if we say that for some mechanical system, states are actually points in a phase-space,  $x \equiv (q, p)$ , and we have a classical Newton law:

$$\frac{d}{dt} x(t) = [x, H]$$

where we use a Poisson bracket formalism, and  $H$  denotes well-known Hamiltonian function in classical mechanics. When we say  $x(t)$  for  $t \in [0, T]$ , we mean an *orbit* in our abstract space (or phase-space). Please note that we can put  $T = \infty$ . So, a definition can be made: dynamics is the evolution of a representative point of phase-space, from a given initial condition  $x_0$  to some later state  $x(t) \equiv x_t$  (eventually  $x_\infty$ ) governed by equations of evolution or equations of motion (Newton equation, Ohm's law, or something we put in). If we are dealing with a continuous variable  $t$  (time), then we are speaking about *flow*. If we took time to be discrete variable  $t_n, n \in \mathbb{N}$  we are dealing with *Discrete dynamics*.

The next useful concept is that of *Lyapunov exponents*. We are examining the divergence of trajectories, whose initial points are close to one another. If we introduce some metric in our space, then we can speak about the distance between our points. Let's limit ourselves only to exponential divergence and let's consider the limit:

$$I = \lim_{t \rightarrow \infty} \ln \frac{d(U_t x_1, U_t x_2)}{d(x_1, x_2)}$$

This is the definition of Lyapunov exponent  $I$ , which denotes how some initially close phase-space points are lying on trajectories which are diverging exponentially. This is the right moment for introducing another concept of fundamental importance. When nearby trajectories are separating fast, we are speaking about *Sensitive Dependence on Initial Conditions (SDIC)*. That means: take an initial condition and take any arbitrary small sphere  $S_\epsilon$  around it, and put some ink into it. Let's now examine the evolution of our 'blob' (our piece of ink) in a limited phase-space. There will exist a time  $t = t(\epsilon)$  when we won't be able to distinguish between the evolved points of our blob. Every point of our initial blob is separating from any other - it's like putting some blue ink into a glass of water and then shaking it. Observe that the concepts of SDIC and Lyapunov exponents are almost the same, except that SDIC is referring only to limited phase space. Now, when we speak about a *chaotic (or stochastic) behavior*, we have to mention that different authors use this word in different senses. What most of them mean is this: *The System is Chaotic if it presents SDIC and if the Lyapunov exponents are positive*. Remains to be said that the exact definition of Chaos is still not available; Chaos is something related to intuition rather than to precise

mathematical definition.

Very illustrative example of chaotic system is given by *logistic map*. Consider the discrete dynamical system (a sequence of real numbers, defined by a recursive relation) :

$$x_{n+1} = c x_n (1 - x_n) \quad , \quad x_0 \in [0,1] \quad , \quad c \in [0,4]$$

If you put some small value for  $c$  , say 1 , you'll see that this sequence has a beautiful limit, it converges to a certain value in  $x$ , let's define it as  $x_\infty$  . If you increase your  $c$  , you'll observe that at certain value your sequence begin to accumulate around two points instead of one as before: there are two different  $x_\infty$  . We say the limit separation or *bifurcation* occurred. Further increasing of parameter  $c$  brings us further bifurcations, limit of sequence separates in four, that in eight limits and so on. You will see that around  $c=3,5699$  a strange thing is happening : there are infinitely many  $x_\infty$  : we entered the region of chaotic behavior. In my opinion a good understanding of this easy example is the best way to understand the notion of Chaos, independently of any mathematical definition. Let's pass now to the main argument of this lecture.

## Entropy

Everyone of us have some more-or-less precise understanding of term "Entropy" ; in thermodynamics we speak about Entropy when we deal with Second Law of Thermodynamics. Entropy characterizes how much energy of our system is not useful ; the sum of free energy and 'entropic energy' is equal to total internal energy of our system. Everyone knows that entropy has one fundamental property : it always increases with time, as a system evolves. It characterizes also how much information is lost during the evolution of our system - if initial conditions have some information , we can say that during the evolution of a thermodynamical system, our information is rendered less precise than initially, until it's completely lost. In psychological terms - thermodynamical system is losing memory about it's initial state. We can say :

$$(\text{information}) = (\text{entropy})^{-1}$$

What's interesting is that the concept of entropy can be introduced into the theory of dynamical systems. It was due to Kolmogorov (1958.) and Sinai (1958.) . There are many definitions of dynamical entropy available ; all of them have the some well-known property : entropy always increases. We will examine two most known : *Topological Entropy (TE)* and *Kolmogorov-Sinai Entropy (KSE)* .

The first one refers to a phase-space  $X$  , which can be left quite general, it's enough to have a topology defined on it. After that we define a *partition of X* , called  $A$  :

$$A \equiv \{a_i\} \quad , \quad X = \bigcup_i a_i \quad , \quad a_i \cap a_j = \emptyset, i \neq j$$

$A$  is a subdivision of  $X$  in it's disjoint parts, at a given initial moment. The number of parts is assumed to be very large. Now let's define a product of two partitions  $A$  and  $B$  :

$$A \vee B \equiv C = \{c_i = a_j \cap b_k \quad \forall j, k\}$$

Now consider a discrete map  $f$  , which operates in iterative way:

$$f(x_0) = x_1 \quad , \quad f(x_1) = f^2(x_0) = x_2 \quad , \quad x_n = f^{n-1}(x_0)$$

between two equal topological spaces  $X$ . Consider now the following quantity which

describes the mixing or ‘shuffling’ of space X :

$$A^n \equiv A \vee f^{-1}A \vee f^{-2}A \vee \dots \vee f^{-n+1}A = \bigvee_{k=0}^{n-1} f^{-k}A$$

Now the definition of TE is as follows :

$$h(f) \equiv \sup_{all A} \left( \lim_{n \rightarrow \infty} \frac{\ln N(A^n)}{n} \right)$$

where  $N(B)$  denotes the number of regions of division B . Obviously , this quantity measures how X is being ‘mixed-up’ by the dynamics f . It takes into account all possible partitionings and it consider a limit for  $n \rightarrow \infty$  which corresponds to  $t \rightarrow \infty$  , in other words TE examines one of the basic properties of chaotic behavior - *mixing of phase-space* . It’s needless to say that the positive  $h(f)$  will indicate a behavior which is not regular in classical sense. The fact that  $A^n$  is defined using backward dynamics instead of forward one, doesn’t make any qualitative difference.

The KSE deals with a particular kind of dynamical systems, so called *mixing systems* . Denote with  $\mathbf{m}$  some invariant measure on X . The measure is *invariant* (under dynamics) if  $\mathbf{m}(A) = \mathbf{m}(f(A))$  ,  $\forall A \subset X$  . Let B and C be two arbitrary regions of X with nonzero measure. Then the dynamics is a mixing if :

$$\lim_{k \rightarrow \infty} \mathbf{m}[f^{-k}(B) \cap C] = \mathbf{m}(B)\mathbf{m}(C)$$

Now let A be a partitioning of X as before , with  $\mathbf{m}(X) = 1$  , an consider a quantity :

$$H_m(A) \equiv - \sum_{i=1}^N \mathbf{m}(a_i) \ln \mathbf{m}(a_i)$$

The definition of KSE is :

$$h_m(f) \equiv \sup_{all C} \left\{ \lim_{n \rightarrow \infty} \frac{H_m(A^n)}{n} \right\}$$

Note that the main difference between TE and KSE is that the former use the measure. Dinaburg (1971.) proved the connection between entropies :

$$h(f) = \sup_m h_m(f)$$

where sup goes over all normalized invariant Borel measures.

KSE gives us a better insight. It is however more difficult to calculate. We can make a simple example of positive entropies, it’s enough to make a simple partitioning of X , we can see that a ‘ordinary’ systems have a zero production of entropy, in the other hand any ‘non-stability’ which would indicate at least a weak presence of chaos have a positive TE and KSE , which integrated over sufficiently large time of evolution, destroys all informational content.

## Conclusions

Non liner phenomena are providing us a deeper look into some properties of Nature. The most beautiful phenomena which has something to do with chaos and bifurcations is life.

It appears that the biological diversity arises from some kind of bifurcations in evolution. If we examine the property of a living organism that it always “*tries to remain alive*”, we’ll see that the living system tends to improve it’s own complexity in order to survive. It appears that one of the ways to achieve this is to increase complexity by successive bifurcations.

There are many other, much more practical, applications of all this; social sciences, economy, engineering, medicine ... Let’s say just that the stock market exchange has some “chaotic” properties, as well as the interaction between populations.

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