Percolation Clustering

Dijana Žilić
Physics Department, Faculty of Science, University of Zagreb, Croatia
July 2000

Abstract

We simulated the random site and the continuum percolation models. The site percolation is considered on a grid of size 50×50. The percolation treshold value p_c and the probability that an occupied site belongs to a spanning cluster P_∞ are determined, using Hoshen-Kopelman algorithm. We obtained the following results: p_c = 0.5899 and P_∞ =0.3746. The continuum percolation in a Swiss cheese model on a square 10x10 was studied. The disk number dependence of disk radius on the percolation treshold is considered. It turns out to be the power law: $n \propto r^{-D}$, where D=(1.94$\pm$0.03). Also, we writed the program which simulates the formation of nanostructures by depositing on a surface particles which diffuse and aggregate.

1. Introduction

Originally percolation theory was presented by S.R.Broadbent and J.M.Hamersley in 1957; they were trying to describe how a fluid spreads through a porous medium. 20 years later, H.E.Stanley showed that the structure which we use to describe percolation is fractal. After 1982., when B.Mandelbrot published the book "The Fractal Geometry of Nature" it became evident that many other phenomena may be described using this theory. Today we see that percolation processes are widespread in nature; they occur in physical, chemical, biological systems and even in metheorology and economy [1] ,[2].

Percolation represents the simplest model of a disordered system. We generally distinguish two different kind of percolation: site and bond percolation.

In the **site percolation** model we have d-dimensional lattice in which the sites are randomly occupied with probability p or empty with probability 1-p. The probability of a site being occupied is independent of that of its neighbors. A cluster is defined as a group of nearest-neighbor sites (nearest neighbor sites are those above, below, left or right of a site).

Fig.1. *Site percolation on the lattice 20×20. Black squares represent occupied sites and white squares represent empty sites. Probability of a site being occupied is p=0.6.*

When the bonds between the sites are randomly occupied, we speak of **bond percolation**. Two occupied bonds belong to the same cluster if they are conected by a path of occupied bonds. In bond percolation, the most common example in physics is a random resistor network, where the metallic wires in a regular network are cut randomly.

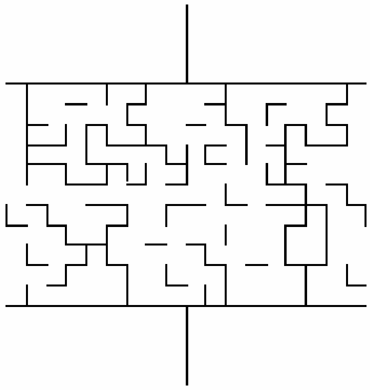


Fig.2. *Bond percolation: random resistor network.*

Also, we can consider **site-bond percolation**, where sites are occupied with probability p and bonds are occupied with probability q. Two occupied sites belong to the same cluster if they are connected by a path of nearest-neighbor occupied sites with occupied bonds in between. For q=1, site-bond percolation reduces to site percolation and for p=1 it reduces to bond percolation.

Physicaly, occupied and empty sites or bonds may stand for very different properties. For example, occupied sites may represent electrical conductors and empty sites represent insulators. Or percolation system may be a combination of superconductors and normal conductors or a combination of magnetic and paramagnetic plates. Percolation system shows different physical properties, depending on the value of probability p. On some critical value p_c for the first time appears the cluster which connects opposite edges of the lattice. In infinite large lattice this cluster is infinite so we call that cluster, even in a finite lattice, infinite cluster. Above probability p_c , which is called percolation treshold or critical concentration, there is infinite cluster; below p_c there are only finite clusters. Concretely, for a lattice which is constructed from conductors and insulators, that means that on p_c electrical current can percolate for the first time from one edge to the other. Below p_c we have an insulator, above p_c we have a conductor. If we have a lattice of superconductors and conductors, on p_c there is a transition from normal-conducting to superconducting phase. Lattice of magnetic and paramagnetic plates on percolation treshold pass over paramagnetic to ferromagnetic phase. But in contrast to the more common thermal phase transitions, where the transition between two phases occurs at a critical temperature, the percolation transition is geometrical phase transition and it is characterised by geometrical features of large clusters in the neighborhood of p_c.

All above-mentioned examples belong to the same class, so-called discrete percolation. However, more natural example of percolation is **continuum percolation** where the positions of the two components of a random mixture are not restricted to the discrete sites of a regular lattice. We can, for example, consider a sheet of conductive material, with circular holes punched randomly in it. By some specific ratio 'fully' and 'empty' that material stops being conductor. This model of percolation we call Swiss cheese model. It has been shown that this model probably belongs to the same universality class as percolation on a lattice, i.e. the critical exponents are the same [3].

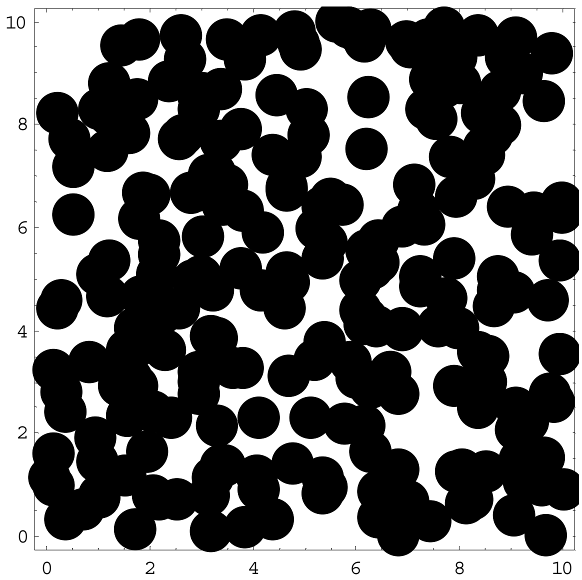


Fig.3. *The Swiss cheese model on the square 10×10.*

2. Numerical methods and results

This work is based on the article [4]. We simulated models of site and continuum percolation in two dimension. For site percolation we determined percolation treshold using Hoshen-Kopelman algorithm. For continuum percolation we studied how, on the percolation treshold, number of holes (disks) depends on their radius. In addition, we simulated the formation of nanostructures by surface depositing particles which diffuse and aggregate. All simulations which we used in this work were made in program Mathematica 4.0.

2.1. Site percolation model

The random site percolation model consists of an m×m random Boolean lattice. This is a lattice in which the sites have values 0 and 1, where 0 represents an empty site and 1 represents an occupied site. We constructed Boolean lattice (fig. 4.) in the following way:

- First, we generated a two-dimensional m×m square lattice; a random value $a \in [0,1]$ is assigned to each site.
- We choose a probability p. If $a \le p$, a is changed to 1; otherwise its value is changed to 0.

$$\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}$$

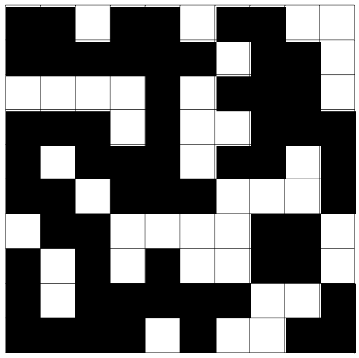


Fig.4. *Boolean lattice 10×10. Black squares (1) represent occupied sites and white squares (0) represent empty sites. Probability of a site being occupied is p=0.6.*

For determining percolation treshold, we used a Hoshen-Kopelman algorithm. All sites in Boolean lattice are labeled in such a way that sites with the same label belong to the same cluster and different labels are assigned to different clusters. If the same label occurs at opposite sides of the system, an infinite cluster exists.

$$\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 3 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 4 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 4 & 4
\end{pmatrix}$$

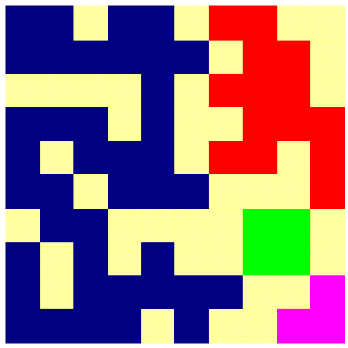


Fig.5. *The lattice from fig.4.: clusters are labeled using Hoshen-Kopelman algorithm.*

In this example we can see that the cluster which is labeled with number 1 (blue cluster) is infinite; other clusters, labeled with 2, 3 and 4 are finite. Because in this lattice we have infinite cluster, that means that percolation treshold p_c is less or equal than our chosen value p=0.6. How accurately we can determine percolation treshold is shown in the following example, for a lattice 6×6.

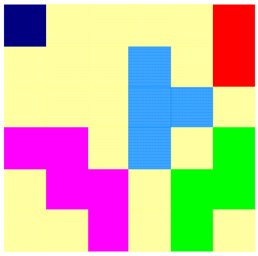
First, we genereted a random lattice 6×6.

$$\begin{pmatrix}
0.0114685 & 0.844083 & 0.479447 & 0.710879 & 0.656348 & 0.198569 \\
0.62817 & 0.448286 & 0.892131 & 0.356938 & 0.505222 & 0.382684 \\
0.518659 & 0.630241 & 0.806216 & 0.328772 & 0.168343 & 0.839653 \\
0.375196 & 0.134779 & 0.943956 & 0.156893 & 0.832743 & 0.153324 \\
0.932487 & 0.312809 & 0.353296 & 0.442446 & 0.276139 & 0.11424 \\
0.725126 & 0.99416 & 0.384008 & 0.757302 & 0.219903 & 0.611476
\end{pmatrix}$$

Subsequently we choose a probability p and construct Boolean lattice. Using Hoshen-Kopelman algorithm we label clusters and look if we have an infinite cluster. For example, we choose two values (lower and higher): p_l=0.4 and p_h=0.7.

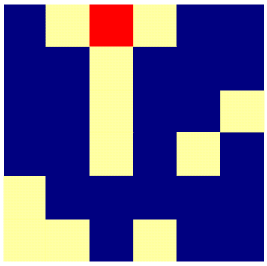
$p_l = 0.4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 4 & 4 & 0 & 3 & 0 & 5 \\ 0 & 4 & 4 & 0 & 5 & 5 \\ 0 & 0 & 4 & 0 & 5 & 0 \end{pmatrix}$$



$p_h = 0.7$

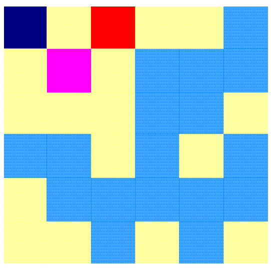
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$



We see that on p_h = 0.7 in lattice exists an infinite cluster and on p_l = 0.4 it doesn't. By gradually decreasing the higher value we approach to the probability at which we don't any more have an infinite cluster and by gradually increasing the lower value we approach a probability at which an infinite cluster appears for the first time. So approaching from the lower and higher side we get a percolation treshold. For our lattice percolation treshold p_c is near $p_c \approx 0.51$

$p = 0.51$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 3 & 3 & 0 & 3 & 0 & 3 \\ 0 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{pmatrix}$$



In this work we studied site percolation on a grid of size 50×50.

Fig.5. *Site percolation on a grid 50×50. Probability of a site being occupied is equal to the critical concentration* p_c*=0.5899.*

We obtained that the percolation treshold, determined on a previously described way is:

$$p_c = 0.5899. \quad (1)$$

That value is very close to the value 0.5927 which we find in a literature [5] for an infinite lattice; relative aberration is 0.5 %.

Total number of sites is 2500. 1487 sites are occupied and distribuited in 88 clusters. Because of so large number of clusters, we couldn't represent a grid with labeling clusters (we couldn't distinguish 88 colors).

The probability P_{∞} that an occupied site belongs to infinite cluster is given by the number of sites in the infinite cluster divided by the total number of occupied sites. In a considered lattice infinite cluster consists of 557 sites. So we got this result:

$$P_{\infty} = 0.3746. \quad (2)$$

2.2. Continuum percolation model

We studied the Swiss cheese model on a square of size 10×10 which is shown in fig. 3. This system is on the percolation treshold when there is one uninterrupted path of disks from one side of the square to the opposite side. We studied how the averege number of disks needed to make that uninterrupted path depends on the size of the disk. We obtained the following results:

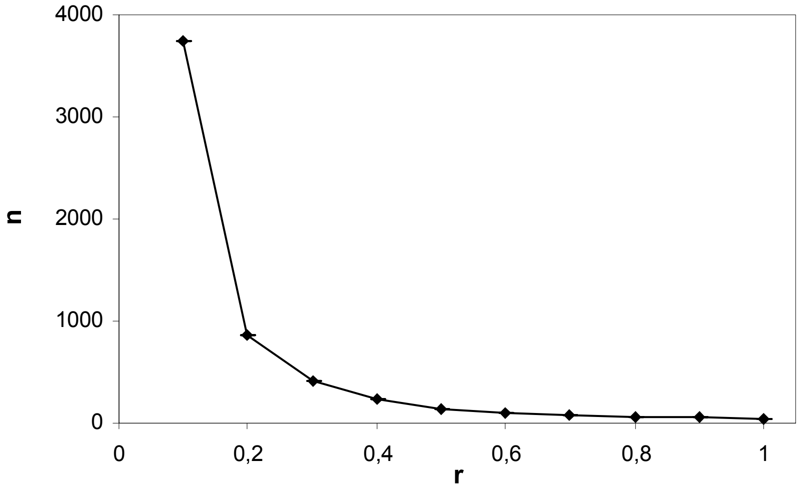


Fig.6. *Number of disks n as a function of disk radius r for a square 10×10, on the percolation treshold.*

If we represent this results in log-log scale, we get:

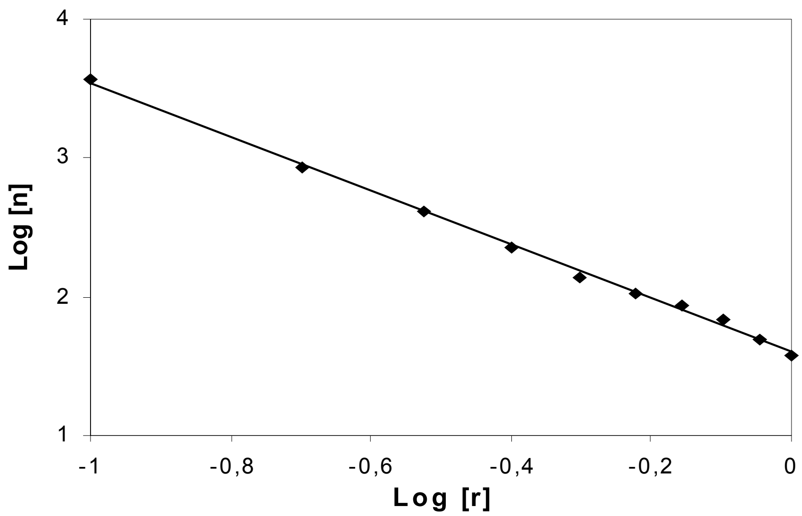


Fig.7. *Number of disks n as a function of disk radius r, represented in log-log scale.*

We see that our data lie approximately on a straight line so we can make a fit to a linear function:

$$\log n = a \log r + b. \qquad (3)$$

We got the following value for parameters a and b:

$$a = -\,1.94 \pm 0.03; \qquad (4)$$
$$b = 1.60 \pm 0.02. \qquad (5)$$

This means that dependence number of disks of a disk radius is going to be a power law:

$$n \sim r^{a}\,. \qquad (6)$$

Because n is found to be proportional to r^{-D}, where D is a fractal dimension, we got that the fractal dimension for a described two-dimensional percolation model has value:

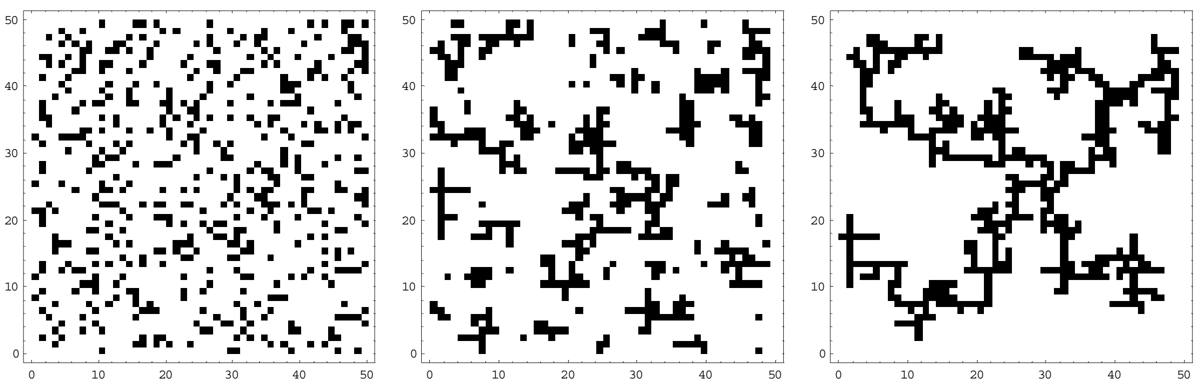
$$D = 1.94 \pm 0.03. \qquad (7)$$

2.3. The formation of nanostructures

We simulated the formation of nanostructures by depositing on a surface particles which diffuse and aggregate. The effects of deposition, diffusion and aggregation can be described in the following way:

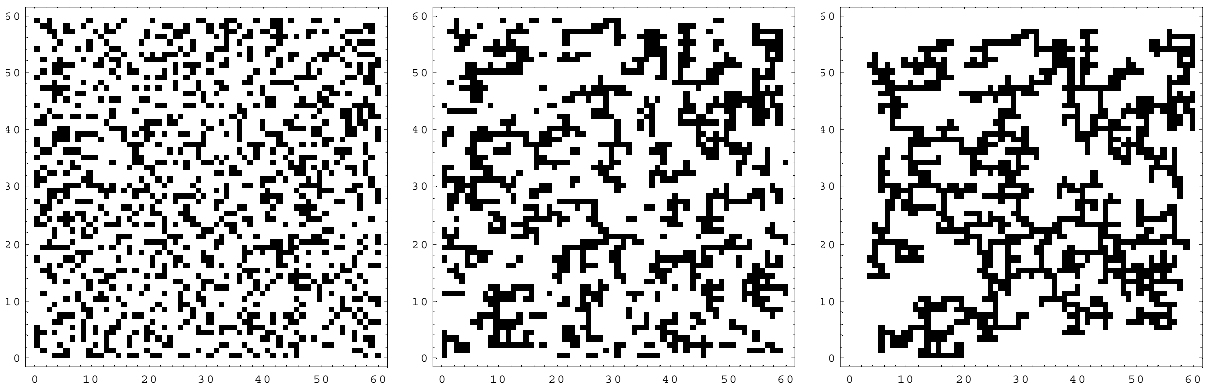
- Deposition: Particles are initially placed randomly on an s×s square lattice.
- Diffusion: At each time step, a randomly selected cluster of connected particles is moved one unit up, down, left or right.
- Aggregation: If particles end up adjacent to one another, their clusters stick together.

The results of simulation on a squares of dimensions 50×50 and 60×60 are shown in the following pictures:



the beginning konfiguration *after 1000 steps* *after 1554 steps*

Fig.8. *Percolation grid 50 x 50; the probability of a site being occupied is p = 0.2.*



the beginning configuration *after 700 steps* *after 1365 steps*

Fig.9. *Percolation grid 60 x 60; the probability of a site being occupied is p = 0.3.*

We can see on these pictures that perfectly random motion can lead to self-organization. The obtaining morphologies resemble experimental images obtained by such low-energy cluster beam deposition experiments on substrates maintained at low temperatures [6].

3. Conclusion

Many systems in nature have no characteristic length or time scale, i.e. they have fractal or, more generally, scalling properties. The range of systems that apparently display power law and hence scale-invariant correlations have increased dramatically in recent years. They are ranging from base pair correlations in DNA, lung inflation and interbeat intervals of the human heart to complex systems involving large numbers of interacting subunits that display ‘free will’, such as city growth, weather fluctuations and even economics [2]. But the fractal properties in different systems, have quite different nature, origin and appearance. In some cases, it is the geometrical shape of on object itself that exhibits obvious fractal features, like above-mentioned nanostructures or “Swiss cheese” objects. However, in the most other cases the fractal properties are more ‘hidden’ and can only be perceived if data are studied as a function of time or mapped onto a graph in some special way.

4. Acknowledgment

The author would like to acknowledge K. Zadro for refering to a literature and for his comments and advices, E. Tafra for his help in the computation, and M. Herak and D. Pajić for their linguistic assistance.

5. Literature

[1] J. Hoshen and R. Kopelman: Phys. Rev. B **14**, 3438 (1976)

[2] S. Havlin, S. V. Buldyrev, A. Bunde, A. L. Goldberger, P. Ch. Ivanov, C.-K. Peng, H. E. Stanley: Physica A **273**, 46-49 (1999)

[3] J. Kertesz: J. Physique **42** L393 (1981)

[4] R. J. Gaylord and P. R. Wellin: Percolation Clustering, in Computer Simulations with Mathematica, Springer-Verlag, New York, 1995

[5] A. Bunde and S. Havlin: Percolation, in Fractals and Disordered Systems, eds. A. Bunde, Springer, Berlin, 1996

[6] P. Jensen, A. Barabasi, H. Larralde, S. Havlin and H. E. Stanley: Nature **368**, 22, March 3, 1994